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If X is a complex projective manifold which carries a contact structure, then the results of [Dem00] and [KPSW00] show that X is either isomorphic to a projectivized tangent bundle of a complex manifold, or that X is Fano and $b_2(X) = 1$. In this paper we study the latter case where X is Fano. It is generally believed that these assumptions imply that X is homogeneous—for an introduction, see the excellent survey in [Bea99].

It follows from our previous work [Keb00] that X can always be covered by lines. Thus, it seems natural to consider the geometry of lines in greater detail. We will show that if $x \in X$ is a general point, then all lines through x are smooth. If $X \not\cong \mathbb{P}_{2n+1}$, then the tangent spaces to these lines generate the contact distribution at x . It follows that the contact structure on X is unique, thus answering a question of C. LeBrun [Leb97, question 11.3]. The result was previously obtained by C. LeBrun [Leb95] if X is a twistor space.

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2. SETUP

2.1. Contact Manifolds. Throughout the present work, let X be a complex projective contact manifold of dimension $2n + 1$. By definition, the contact structure is given by a sequence of vector bundles

$$0 \longrightarrow F \longrightarrow T_X \xrightarrow{\theta} L \longrightarrow 0$$

where L is of rank 1 and the contact form $\theta \in H^0(\Omega_X^1 \otimes L)$ yields a nowhere vanishing section

$$\theta \wedge (d\theta)^{\wedge n} \in H^0(K_X \otimes L^{n+1})$$

—it is an elementary calculation to see that $\theta \wedge (d\theta)^{\wedge k}$ is a well-defined section of $\Omega_X^{2k+1} \otimes L^{k+1}$ for all numbers $n \geq k \geq 1$. In particular, we assume that $-K_X = (n+1)L$.

The assumptions imply that the natural map $[\cdot, \cdot] : F \otimes F \rightarrow L$ derived from the Lie-bracket is non-degenerate. In view of the Frobenius theorem this means that if $Y \subset X$ is an F -integral submanifold, i.e. one where $T_Y \subseteq F|_Y$, then $\dim Y \leq n$. If Y is of maximal dimension $\dim Y = n$, then Y is called “Legendrian”. Note that some authors (e.g. [Hwa97], [KPSW00]) prefer to use “Lagrangian” instead of “Legendrian”.

The usual Darboux theorem of real contact and symplectic geometry applies equally well in the complex case. Thus, for any point $x \in X$ we can find coordinates $(z_i)_{i=1\dots 2n+1}$ centered about x and a bundle coordinate for L so that we can write

$$\theta = dz_{2n+1} + \sum_{i=1\dots n} z_i dz_{n+i}$$

In particular, we remark the following:

Remark 2.1. If $x \in X$ is any point and $\vec{v} \in T_{X,x}$ any tangent vector, then there exists a Legendrian submanifold $U \subset X$ which contains x and is transversal to \vec{v} , i.e. $\vec{v} \notin T_U|_x$.

2.2. Parameter spaces. For the benefit of readers coming from differential geometry we will briefly recall some facts about the parameter spaces which we will use in the sequel. Our chief reference will be [Kol96, chap. II], and our notation will be compatible with this book. Mori’s paper [Mor79] on the Hartshorne conjecture is also recommended for these matters.

If V is any projective manifold and C a projective variety, then we will often parameterize those morphisms from C to V which are birational onto their images. In fact, there exists a scheme $\text{Hom}_{\text{bir}}(C, V)$ whose geometric points correspond to these morphisms. Furthermore, there exists a “universal morphism”: $\mu : \text{Hom}_{\text{bir}}(C, V) \times C \rightarrow V$. We refer to [Kol96, chapt. II.1] for an authoritative reference on this.

The tangent space to $\text{Hom}_{\text{bir}}(C, V)$ is described as follows: If a birational morphism $f : C \rightarrow V$ is given, then the (Zariski-)tangent space of $\text{Hom}_{\text{bir}}(C, V)$ at f corresponds naturally to sections in $H^0(C, f^*(T_V))$.

If $c \in C$ and $v \in V$ are (geometric) points, the subfamily of morphisms mapping c to v is usually denoted as $\text{Hom}_{\text{bir}}(C, V, c \mapsto v)$. The tangent space to $\text{Hom}_{\text{bir}}(C, V, c \mapsto v)$ at a point f corresponds to $H^0(C, f^*(T_V) \otimes \mathcal{I}_c)$, where \mathcal{I}_c is the ideal sheaf of c .

In the special case that $C \cong \mathbb{P}_1$, and $f \in \text{Hom}_{bir}(\mathbb{P}_1, V)$ a morphism whose image contains a point $v \in X$, the Riemann-Roch theorem yields an estimate for the dimensions of the deformation spaces

$$(2.1) \quad \begin{aligned} \dim_{[f]} \text{Hom}_{bir}(\mathbb{P}_1, V) &\geq -K_V \cdot \ell + \dim V \\ \dim_{[f]} \text{Hom}_{bir}(\mathbb{P}_1, V, [0 : 1] \mapsto v) &\geq -K_X \cdot \ell \end{aligned}$$

where $\ell := \text{Image}(f)$. See [Kol96, prop. II.1.13 and thm. II.1.7] for an explanation and a proof.

The group $\mathbb{P}SL_2$ acts on the normalization $\text{Hom}_{bir}^n(\mathbb{P}_1, V)$ of $\text{Hom}_{bir}(\mathbb{P}_1, V)$, and the geometric quotient in the sense of Mumford [FM82] exists, see [Mor79, lem. 9]. More precisely, by [Kol96, thm. II.2.15] there exists a commutative diagram

$$(2.2) \quad \begin{array}{ccccc} & & \mu & & \\ & \text{Hom}_{bir}^n(\mathbb{P}_1, V) \times \mathbb{P}_1 & \xrightarrow{U} & \text{Univ}^{rc}(V) & \xrightarrow{\iota} X \\ & \downarrow & & \downarrow \pi & \\ & \text{Hom}_{bir}^n(\mathbb{P}_1, V) & \xrightarrow{u} & \text{RatCurves}^n(V) & \end{array}$$

where u and U are principal $\mathbb{P}SL_2$ bundles, π is a \mathbb{P}_1 -bundle and the restriction of ι to any fiber of π is a morphism which is birational onto its image. We call the quotient space $\text{RatCurves}^n(V)$ the “parameter space of rational curves on V ”. The letter “ n ” in RatCurves^n may be a little confusing. It has nothing to do with the dimension of V , but serves as a reminder that the parameter space is isomorphic to the normalization of a suitable quasiprojective subset of the Chow-variety.

2.3. Lines. Unless otherwise mentioned, throughout this work we will assume that X is Fano and that $b_2(X) = 1$. In this setup it follows from the classic work of Mori ([Mor79], but see also [CKM88, lect. 1]) that we can find an irreducible component $H \subset \text{RatCurves}^n(X)$ with the following properties:

1. the evaluation morphism ι is dominant
2. if $x \in X$ is a general point, then the subfamily $H_x := \pi(\iota^{-1}(x)) \subset H$ (i.e. the subfamily which parameterizes curves containing x) is compact.
3. if $\ell \subset X$ is a curve which is associated with a point of H , then $1 \leq -K_X \cdot \ell \leq \dim X + 1$.

Since X is a contact manifold, we have that $-K_X = (n+1)L$, and it follows from point (3) that either $L \cdot \ell = 2$ or $L \cdot \ell = 1$.

If $L \cdot \ell = 2$, then the estimate (2.1) shows that

$$\dim \text{Hom}_{bir}(\mathbb{P}_1, X, [0 : 1] \mapsto x) \geq -K_X \cdot \ell = \dim X + 1.$$

Because there is a 2-dimensional group of automorphisms of \mathbb{P}_1 which fix $[0 : 1]$, the assumption that $L \cdot \ell = 2$ implies

$$\dim H_x \geq \dim X - 1,$$

and it follows from the generalized Kobayashi-Ochiai theorem of [Keb00, thm. 3.6] or [KS99, thm. 0.2] that $X \cong \mathbb{P}_{2n+1}$. For that reason we assume in the sequel that $L \cdot \ell = 1$. We call ℓ a “contact line”.

Remark 2.2. The assumption that $L \cdot \ell = 1$ implies that the irreducible component $H \subset \text{RatCurves}^n(X)$ is compact. See [Kol96, prop. II.2.14].

Remark 2.3. Let $f : \tilde{\ell} \rightarrow \ell$ be the normalization. Since $\tilde{\ell} \cong \mathbb{P}_1$, and $T_{\tilde{\ell}} \cong \mathcal{O}_{\mathbb{P}_1}(2)$, it is clear that the map $T_{\tilde{\ell}} \rightarrow f^*(L)$ must be trivial. It follows that $T_{\tilde{\ell}}|_x \subset F|_x$ for all smooth points $x \in \ell$. We say that contact lines are F -integral where they are smooth.

3. DEFORMATIONS OF LINES

We will show that lines passing through a general point are smooth. For this, we employ deformations of lines in order to obtain sections of L . The following local proposition shows that there are severe restrictions for such sections to exist.

Proposition 3.1. *Consider a family of morphisms $\Phi_t : \Delta_C \rightarrow X$ written as*

$$\begin{aligned} \Phi : \Delta_{\mathcal{H}} \times \Delta_C &\rightarrow X \\ (t, z) &\mapsto \Phi_t(z) \end{aligned}$$

where $\Delta_{\mathcal{H}}$ and Δ_C are unit disks. Assume that for all $t \in \Delta_{\mathcal{H}}$ the image $\Phi_t(\Delta_C)$ is F -integral, i.e. assume that $\Phi^*(\theta) \left(\frac{\partial}{\partial z} \right) \equiv 0$. If

$$\sigma = \Phi_0^*(\theta) \left(\frac{\partial}{\partial t} \right) \in H^0(\Delta_C, \Phi_0^*(L))$$

vanishes at $0 \in \Delta_C$ but does not vanish identically, then σ vanishes with multiplicity at least two if and only if

$$(3.1) \quad \Phi^*(d\theta) \left(\frac{\partial}{\partial t} \Big|_{(0,0)}, \frac{\partial}{\partial z} \Big|_{(0,0)} \right) = 0.$$

Remark 3.2. The exterior derivative $d\theta$ which appears in equation (3.1) depends on a choice of bundle coordinates for L and is therefore not well-defined. Note, however, that the requirement $\sigma(0) = 0$ implies that $T\Phi \left(\frac{\partial}{\partial t} \Big|_{(0,0)} \right) \in F|_{\Phi(0,0)}$, where $T\Phi$ is the tangent map associated with Φ . In this setting an elementary calculation shows that the validity of equation (3.1) does in fact not depend on the choice of bundle coordinates.

Proof. Recall the following formula: if ω is any 1-form on a manifold, and \vec{X}_0 and \vec{X}_1 are vector fields, then

$$(3.2) \quad d\omega(\vec{X}_0, \vec{X}_1) = \vec{X}_0(\omega(\vec{X}_1)) - \vec{X}_1(\omega(\vec{X}_0)) - \omega([\vec{X}_0, \vec{X}_1]).$$

See e.g. [War71, prop. 2.25(e) on p. 70] for an explanation. We choose local bundle coordinates on $\Phi^*(L)$ and apply equation (3.2) with $\omega = \Phi^*(\theta)$, $\vec{X}_0 = \frac{\partial}{\partial z}$ and $\vec{X}_1 = \frac{\partial}{\partial t}$. Since $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial z}$ commute and $\Phi^*(\theta) \left(\frac{\partial}{\partial z} \right) \equiv 0$, we obtain

$$\frac{\partial}{\partial z} \Phi^*(\theta) \left(\frac{\partial}{\partial t} \right) = d\Phi^*(\theta) \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial z} \right).$$

Note that $d\Phi^*(\theta) = \Phi^*(d\theta)$ and evaluate at $t = 0, z = 0$. □

Another argument which uses the deformation of lines shows that most lines are smooth.

Proposition 3.3. *If $x \in X$ is a general point and ℓ a contact line passing through x , then ℓ is smooth.*

Proof. Assume to the contrary, i.e. assume that for a general point $x \in X$ there exists a singular line ℓ passing through x . Recall that the rational curve ℓ can always be dominated by an integral singular plane cubic, i.e. by a rational curve with a single node or cusp. We will reach a contradiction by constructing a section of the pull-back of L to the plane cubic which vanishes at a prescribed generically chosen point. This section will be constructed by a deformation of the singular curves.

Because x was chosen to be general, there exists a singular (i.e. cuspidal or nodal) plane cubic $C \subset \mathbb{P}_2$ and an irreducible component $\mathcal{H} \subset \text{Hom}^{\text{birat}}(C, X)$ such that the universal morphism $\mu : \mathcal{H} \times C \rightarrow X$ is dominant and such that for all $f \in \mathcal{H}$ we have $\deg f^*(L) = 1$.

Fix a general morphism $f \in \mathcal{H}$ and note that there is an open set $U \subset C$ such that for all $c \in U$, the tangent map of the restricted morphism $\mu_c := \mu|_{\mathcal{H} \times \{c\}}$ has maximal rank at f :

$$\text{rank}_{[f]} T\mu_c = \dim X = 2n + 1.$$

Recall from [Har77, II.6.10.2, II.6.11.4 and Ex. II.6.7] that the smooth points of C are in 1:1-correspondence with line bundles of degree one, and fix a point $c \in U$ such that $\mathcal{O}_C(c) \not\cong f^*(L)$.

Next, let $U_X \subset X$ be a Legendrian submanifold of X which contains x and is transversal to $f(C)$ at x . By remark 2.1, these exist in abundance. Since μ_c has maximal rank, we can find a section $U_{\mathcal{H}} \subset \mathcal{H}$ over U_X , i.e. a submanifold $U_{\mathcal{H}}$ such that $\mu_c|_{U_{\mathcal{H}}} : U_{\mathcal{H}} \rightarrow U_X$ is an isomorphism. By construction, $\mu(U_{\mathcal{H}} \times C)$ has dimension $n + 1$ and cannot be Legendrian. It follows that there exists a unit disc $\Delta_{\mathcal{H}} \subset U_{\mathcal{H}}$ with coordinate t centered about $f = \{t = 0\}$ such that

$$\sigma := f^*(\theta) \left(\frac{\partial}{\partial t} \Big|_{t=0} \right) \in H^0(C, f^*(L)) \setminus \{0\}.$$

For this, recall that the tangent vector $\frac{\partial}{\partial t}|_{t=0} \in T_{\mathcal{H}}|_f$ is canonically identified with an element in $H^0(C, f^*(T_X))$. By choice of $U_{\mathcal{H}}$, we have $\sigma(c) = 0$. By choice of c , this is impossible, a contradiction. \square

It is an immediate corollary that a tangent morphism exists.

Corollary 3.4. *If $x \in X$ is a general point, then there exists a tangent morphism $\tau_x : H_x \rightarrow \mathbb{P}(F^*|_x)$ which sends a line ℓ to its tangent space $T_{\ell}|_x \subset F|_x$.*

The morphism τ_x was already studied in [Hwa97]. It was, however, not all clear at that time that τ_x really is a morphism and not just a rational map. See [Keb00] for a weaker but more general result.

Finally, we remark that deformations of lines through a general point are unobstructed in a strong sense.

Lemma 3.5. *If $x \in X$ is a general point and ℓ any line through x , then*

$$T_X|_{\ell} \cong \mathcal{O}_{\ell}(2) \oplus \mathcal{O}_{\ell}(1)^{\oplus n-1} \oplus \mathcal{O}_{\ell}^{\oplus n+1}.$$

Proof. It follows from the definition of the contact structure that $F \cong F^* \otimes L$. Since $L|_{\ell} \cong \mathcal{O}_{\ell}(1)$, and since vector bundles on \mathbb{P}_1 always decompose into sums of line bundles we may therefore write

$$F|_{\ell} \cong \bigoplus_{i=1}^n (\mathcal{O}_{\ell}(a_i) \oplus \mathcal{O}_{\ell}(1 - a_i))$$

where $a_i > 0$. Thus, the splitting of $F|_\ell$ has exactly n positive entries. It follows that the splitting of $T_X|_\ell$ has at most n positive entries. By [KMM92, prop. 1.1], $T_X|_\ell$ is nef, and since $c_1(T_X|_\ell) = n + 1$, the claim follows. \square

We will apply lemma 3.5 to study the locus of lines through a general point. For this, fix a general point $x \in H$, define the subfamily $H_x \subset H$ as in section 2.3 and consider the restricted diagram associated to diagram 2.2 on page 3.

$$\begin{array}{ccc} \tilde{U}_x & \xrightarrow{\tilde{\iota}_x} & \text{locus}(H_x) \subset X \\ \tilde{\pi}_x \downarrow & & \\ \tilde{H}_x & \xrightarrow{\tilde{\tau}_x} & \mathbb{P}(F^*|_x) \end{array}$$

Here \tilde{H}_x is the normalization of H_x , \tilde{U}_x the normalization of the pull-back $\text{Univ}^{rc}(X) \times_{\text{RatCurves}^n(X)} \tilde{H}_x$ and $\text{locus}(H_x) = \iota(\pi^{-1}(H_x))$. Recall from remark 2.2 that H and therefore H_x are compact. In particular, $\text{locus}(H_x)$ is a proper subvariety of X .

It follows from [Kol96, thms. II.3.11.5 and II.2.8] that \tilde{H}_x is smooth and $\tilde{\pi}_x$ a \mathbb{P}_1 -bundle. As an immediate corollary to the preceding lemma, we obtain that both $\tilde{\iota}_x$ and $\tilde{\tau}_x$ are immersive.

Corollary 3.6. *If $x \in X$ is a general point, then*

1. *the universal morphism $\tilde{\iota}_x : \tilde{U}_x \rightarrow \text{locus}(H_x) \subset X$ is a birational immersion away from a section σ_∞ which is contracted to x .*
2. *the tangent map $\tilde{\tau}_x : \tilde{H}_x \rightarrow \mathbb{P}(F^*|_x)$ is also an immersion*

Proof. The fact that $\tilde{\iota}_x$ and $\tilde{\tau}_x$ are immersive follows from [Kol96, props. II.3.4 and II.3.10] and lemma 3.5. It follows from an argument of Miyaoka that $\tilde{\iota}_x$ is birational because all lines through x are smooth. For this, see [Kol96, prop. V.3.7.5]. \square

4. LINES THROUGH A FIXED POINT

For a better understanding of contact Fano manifolds, the locus of lines through a given point is of greatest interest. The following proposition gives a first description. This result is contained implicitly in [KPSW00, sect. 2] and we could have used the results of that paper here, but we prefer to give a short and self-contained proof in our context.

Proposition 4.1. *If $x \in X$ is any point, then $\text{locus}(H_x)$ has dimension n and is F -integral where it is smooth.*

Proof. Since $-K_X \cdot \ell = n + 1$, it follows from the estimate (2.1) for the dimension of the parameter space that

$$\dim \text{Hom}_{\text{bir}}(\mathbb{P}_1, X, [0 : 1] \mapsto x) \geq n + 2.$$

By Mori's bend-and-break argument [Kol96, thm. II.5.4], for a given point $y \in X \setminus \{x\}$, there are at most finitely many lines containing both x and y . It follows that

$$\dim \text{locus}(H_x) = \dim \text{Hom}(\mathbb{P}_1, X, [0 : 1] \mapsto x) - \dim \text{Aut}(\mathbb{P}_1, [0 : 1]) \geq n.$$

Claim. The subvariety $\text{locus}(H_x)$ is F -integral where it is smooth.

Application of the claim. It follows immediately from Frobenius' theorem and from the non-degeneracy of the contact distribution that $\dim \text{locus}(H_x) \leq n$, and we are done.

Proof of the claim. Let $y \in \text{locus}(H_x)$ be a general (smooth) point, $\ell \in H_x$ a curve which contains x and y and is smooth at y . By general choice of y , such a curve can always be found. Let $f : \mathbb{P}_1 \rightarrow \ell$ be a birational morphism with $f([0 : 1]) = x$ and $f([1 : 1]) = y$. If

$$\mathcal{H} \subset \text{Hom}^{\text{birat}}(\mathbb{P}_1, X, [0 : 1] \mapsto x)$$

is an irreducible component of the reduced Hom-scheme which contains f , then we have that

$$(4.1) \quad T_{\text{locus}(H_x)}|_y \subset T_\ell|_y + \text{Image}(T_{\mathcal{H}}|_f \rightarrow T_X|_y)$$

where $T_{\mathcal{H}}|_f$ is identified with $H^0(\mathbb{P}_1, f^*(T_X) \otimes \mathcal{O}_{\mathbb{P}_1}(-[0 : 1]))$ and the map $T_{\mathcal{H}}|_f \rightarrow T_X|_y$ is an application of the tangent map Tf and evaluation at y . In other words, the tangent space $T_{\text{locus}(H_x)}|_y$ at y is spanned by the tangent space to the curve ℓ and by sections of $f^*(T_X)$ which vanish at $[0 : 1]$. We refer to [Kol96, prop. II.3.4] for a proof of (4.1).

In remark 2.3 we have already seen that $T_\ell \subset F|_\ell$ so that it suffices to prove that

$$\text{Image}(T_{\mathcal{H}}|_f \rightarrow T_X|_y) \subset F|_y$$

In other words, we have to show that if $\Delta_{\mathcal{H}} \subset \mathcal{H}$ is any unit disk centered about f with coordinate t , then the section $\sigma \in H^0(\mathbb{P}_1, f^*(T_X))$ associated with $\frac{\partial}{\partial t}|_{t=0}$ is contained in $H^0(\mathbb{P}_1, f^*(F))$. For this, note that proposition 3.1 asserts that the section $f^*(\theta)(\sigma) \in H^0(\mathbb{P}_1, f^*(L))$ has a zero at $0 \in \mathbb{P}_1$ whose order is at least two. But since $\deg f^*(L) = 1$, this implies that $f^*(\theta)(\sigma)$ vanishes identically. In particular, $\sigma \in H^0(\mathbb{P}_1, f^*(F))$.

This proves that $T_{\text{locus}(H_x)}|_y \subset F|_y$. Since y was chosen generically, $\text{locus}(H_x)$ is F -integral where it is smooth, and we are done. \square

Under the assumptions spelled out in section 2, if $x \in X$ is a general point, then the contact distribution $F|_x$ is generated by the tangent spaces to lines through x . Hence, it is canonically given. Before starting the proof, however, it is convenient to introduce the following notation first.

Notation 4.2. Consider the incidence variety

$$V := \{(x', x'') \in X \times X \mid x'' \in \text{locus}(x')\} \subset X \times X.$$

An elementary calculation shows that V is a closed subvariety of $X \times X$. We call V the “universal locus of lines through points”.

Let $\pi_1, \pi_2 : V \rightarrow X$ are the projection morphisms. Then for every $x \in X$ we have that (set-theoretically) $\pi_2(\pi_1^{-1}(x)) = \text{locus}(H_x)$. It may well happen that $\pi_1^{-1}(x)$ is not reduced for special points $x \in X$.

If $Y \subset X$ is a subset, we shall write $V|_Y$ for $\pi_1^{-1}(Y)$.

Lemma 4.3. *Let $V \subset X \times X$ be the universal locus of lines through points which we defined above. Let Δ be a unit disk with coordinate t and $\gamma : \Delta \rightarrow X$ an embedding. Then there exists an open set $V^0 \subset V|_{\gamma(\Delta)}$ such that $\pi_2(V^0)$ is a submanifold of dimension*

$$\dim \pi_2(V^0) = n + 1.$$

In particular, by Frobenius' theorem, $\pi_2(V^0)$ is not F -integral.

Proof. We have already seen in proposition 4.1 that $\pi_1|_V$ is equidimensional of relative dimension n . Thus, V is a well-defined family of algebraic cycles over X in the sense of [Kol96, I.3.10] and the universal property of the Chow-variety yields a map $\phi : X \rightarrow \text{Chow}(X)$ such that V is the pull-back of the universal family over $\text{Chow}(X)$. Because $\dim \text{locus}(H_x) = n < \dim X$, it is clear that the image of ϕ is not a point. Use the assumption that $b_2(X) = 1$ to obtain that ϕ is actually a finite morphism. Because two reduced algebraic cycles are equal if and only if their supports agree, it follows that for a given point $x_0 \in X$, there are at most finitely many points $(x_i)_{i=1\dots k} \subset X$ such that

$$\text{locus}(H_{x_0}) = \text{locus}(H_{x_i}).$$

In particular, if $V^0 \subset V|_{\gamma(\Delta)}$ is an open set such that $\pi_2|_{V^0}$ is an embedding, then $\pi_2(V^0)$ has dimension $n + 1$. Hence the claim. \square

With these preparations we can now start the proof of the main theorem of this work.

Theorem 4.4. *If $x \in X$ is a general point, then $F|_x$ is spanned by the image of the tangent map τ_x .*

Proof. Our argument involves an analysis of the deformations of $\text{locus}(H_y)$ which are obtained by varying the base point y . We shall argue by contradiction and assume that the assertion of the proposition is wrong. With this assumption we will construct a family of morphisms $\mathbb{P}_1 \rightarrow X$ which contradicts proposition 3.1, and we are done.

We will now produce a map γ to which lemma 4.3 can be applied. Assuming that the statement of the proposition is wrong, we can find an analytic open neighborhood $U = U(x) \subset X$ and a subbundle $F' \subset F|_U$ such that

1. For all $y \in U$, the vector spaces $F'|_y$ and $\text{Span}(\text{Image } \tau_y) \subset F|_y$ are perpendicular with respect to the non-degenerate form $F \otimes F \rightarrow L$ which comes with the contact structure.
2. All lines through U are smooth.

After shrinking U , if necessary, let $\vec{v} \in H^0(U, F')$ be a nowhere-vanishing vector field. Thus, if $y \in U$ is any point and $\ell \ni y$ is any line through y , then

$$(4.2) \quad T_\ell|_y \subset \vec{v}(y)^\perp,$$

where “ \perp ” again means: perpendicular with respect to the non-degenerate form on F . Let Δ be a unit disc with coordinate t and $\gamma : \Delta \rightarrow X$ be an integral curve of \vec{v} with $\gamma(0) = x$.

Now let $\mathcal{H} \subset \text{Hom}_{\text{bir}}(\mathbb{P}_1, X)$ be the family of morphisms parameterizing the curves associated with H . Set

$$\mathcal{H}_\Delta := \{f \in \mathcal{H} \mid f([0 : 1]) \in \gamma(\Delta)\}.$$

If $\mu_\Delta : \mathcal{H}_\Delta \times \mathbb{P}_1 \rightarrow X$ is the universal morphism, then it follows by construction that

$$\mu_\Delta(\mathcal{H}_\Delta \times \mathbb{P}_1) = \pi_2(V|_{\gamma(\Delta)}) \supset \pi_2(V^0).$$

In particular, since $\pi_2(V^0)$ is not F -integral, for a general point $(f, p) \in \mathcal{H}_\Delta \times \mathbb{P}_1$ there exists a tangent vector $\vec{w} \in T_{\mathcal{H}_\Delta \times \mathbb{P}_1}|_{(f, p)}$ such that the image of the tangent

map is not in F :

$$T\mu_\Delta(\vec{w}) \notin F$$

Decompose $\vec{w} = \vec{w}' + \vec{w}''$, where $\vec{w}' \in T\mathbb{P}_1|_p$ and $\vec{w}'' \in T\mathcal{H}_\Delta|_f$. Then, since $f(\mathbb{P}_1)$ is F -integral, it follows that $T\mu_\Delta(\vec{w}') \in F$ and therefore

$$(4.3) \quad T\mu_\Delta(\vec{w}'') \notin F.$$

As a next step, choose an immersion

$$\begin{array}{ccc} \beta: & \Delta & \rightarrow \mathcal{H}_\Delta \\ & t & \mapsto \beta_t \end{array}$$

such that $\beta_0 = f$ and such that

$$T\beta \left(\frac{\partial}{\partial t} \Big|_{t=0} \right) = \vec{w}''$$

In particular, if $\sigma \in H^0(\mathbb{P}_1, f^*(T_X))$ is the section associated with $\vec{w}'' = T\beta(\frac{\partial}{\partial t}|_{t=0})$, and $\sigma' := f^*(\theta)(\sigma) \in H^0(\mathbb{P}_1, f^*(L))$, then the following holds:

1. it follows from (4.3) and from [Kol96, prop. II.3.4] that σ' is not identically zero.
2. at $[0 : 1] \in \mathbb{P}_1$, the section σ satisfies $\sigma([0 : 1]) \in f^*(T_{\gamma(\Delta)}) \subset f^*(F)$. In particular, $\sigma'([0 : 1]) = 0$.
3. If z is a local coordinate on \mathbb{P}_1 about $[0 : 1]$, then it follows from (4.2) that $\frac{\partial}{\partial z}|_{[0:1]} \in f^*(F)$ and $\sigma([0 : 1]) \in f^*(F')$ are perpendicular with respect to the non-degenerate form.

Items (2) and (3) ensure that we can apply proposition 3.1 to the family β_t . Since the section σ' does not vanish completely, the proposition states that σ' has a zero of order at least two at $[0 : 1]$. But σ' is an element of $H^0(\mathbb{P}_1, f^*(L))$, and $f^*(L)$ is a line bundle of degree one. We have thus reached a contradiction, and the proof of theorem 4.4 is therefore finished. \square

It follows that there are only two types of contact manifolds whose structure is not unique.

Corollary 4.5. *If X is any complex projective contact manifold with more than one contact structure, then either $X \cong \mathbb{P}_{2n+1}$ or $X \cong \mathbb{P}(T_Y)$ for a manifold Y whose tangent bundle T_Y has bundle automorphisms.*

We refer to [KPSW00, sect. 2.6] for a study of the different contact structures on $\mathbb{P}(T_Y)$.

Proof. By [KPSW00, prop. 3.1], the canonical bundle ω_X is not nef. But then it has already been shown in [KPSW00, thm. 1.1] that X is automatically of type $\mathbb{P}(T_Y)$ if $b_2(X) > 1$. We may therefore assume without loss of generality that X is Fano and that $b_2(X) = 1$.

Let $x \in X$ be a general point, and $\ell \ni x$ a minimal rational curve through x . It follows from the classical argument of Mori that $-K_X \cdot \ell \leq \dim X + 1$. Since X is a contact manifold, $-K_X = (n+1)L$ so that $L \cdot \ell \in \{1, 2\}$. If $L \cdot \ell = 2$, then $X \cong \mathbb{P}_{2n+1}$. If $L \cdot \ell = 1$, then theorem 4.4 shows that the contact distribution F is canonically defined, hence unique. \square

REFERENCES

- [Bea99] A. Beauville. Riemannian holonomy and algebraic geometry. LANL-Preprint math.AG/9902110, 1999.
- [CKM88] H. Clemens, J. Kollár and S. Mori. Higher Dimensional Complex Geometry. *Astérisque*, 16, 1988.
- [Dem00] J.P. Demailly. On the Frobenius integrability of certain holomorphic p -forms. LANL-Preprint math.AG/0004067, 2000.
- [FM82] J. Fogarty and D. Mumford. *Geometric Invariant Theory*, 2nd edition, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer, 1982.
- [Har77] R. Hartshorne. *Algebraic Geometry*, volume 52 of *Graduate Texts in Mathematics*. Springer, 1977.
- [Hwa97] J.-M. Hwang. Rigidity of homogeneous contact manifolds under Fano deformation. *J. Reine Angew. Math.*, 486:153-163, 1997
- [KS99] Y. Kachi and E. Sato. Polarized varieties whose points are joined by rational curves of small degrees. *Illinois J. Math.*, 43(2):350–390, 1999.
- [Keb00] S. Kebekus. Families of singular rational curves. LANL-Preprint math.AG/0004023, 2000.
- [KPSW00] S. Kebekus, T. Peternell, A. Sommese, and J. Wiśniewski. Projective contact manifolds. LANL-Preprint math.AG/9810102. To appear in *Invent. Math.*, 2000.
- [Kol96] J. Kollár. *Rational Curves on Algebraic Varieties*, volume 32 of *Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge*. Springer, 1996.
- [KMM92] J. Kollár, Y. Miyaoka and S. Mori. Rational Connectedness and Boundedness for Fano Manifolds. *J. Diff. Geom.* 36:765–769, 1992.
- [Leb95] C. LeBrun. Fano Manifolds, Contact Structures and Quaternionic Geometry. *Int. Journ. of Maths.* 3:419–437, 1995.
- [Leb97] C. LeBrun. Twistors for tourists: A pocket guide for algebraic geometers. Kollar, J. (ed.) et al., *Algebraic geometry. Proceedings of the Summer Research Institute, Santa Cruz, CA, USA, 1995. Proc. Symp. Pure Math.* 62(2):361–385, 1997.
- [Mor79] S. Mori. Projective Manifolds with ample tangent bundles. *Ann. Maths.* 110:593–606, 1979.
- [War71] F. Warner. *Foundations of Differentiable Manifolds and Lie Groups*. Scott, Foresman and Company, Glenview, Illinois and London, 1971.

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